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ON THE STRUCTURE OF MINIMUM-WEIGHT k-CONNECTED SPANNING NETWORKS*

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Abstract. The problem of finding a minimum-weight k-connected spanning subgraph of a complete graph, assuming that the edge weights satisfy the triangle inequality, is studied. It is shown that the class of minimum-weight k-edge connected spanning subgraphs can be restricted to those subgraphs which, in addition to the connectivity requirements, satisfy the following two conditions:

(I) Every vertex has degree k or $k + 1$;

(II) Removing any 1, 2, $\cdots$, or k edges does not leave the resulting connected components all k-edge connected.

For the k-vertex connected case, the parallel result is obtained with "k-edge" replaced by "k-vertex," with the added technical restriction that $|V| \geqq 2k$ for condition (I) to hold. This generalizes recent work of Monma, Munson, and Pulleyblank for the case $k = 2$.

Key words. survivability, graph theory, lifting, connectivity

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1. Introduction. In the design of communication or transportation networks, it is frequently important to produce networks of low "cost" which are also "survivable." In many cases the cost arises, to a good degree of approximation, in the form of edge weights that satisfy the "triangle inequality" (defined in precise form below). The overall cost, or weight, or a network is the sum of the individual edge weights. For survivability reasons, the network must satisfy certain connectivity requirements (see [CW], [GM], [MS], [SWK] for more motivation). A typical survivability requirement is that the removal of any $(k - 1)$ or fewer edges (or vertices) leaves the remaining network connected. The following standard definitions are required to make the above statements precise.

A *graph* or *network* $G = (V, E)$ is called *k-edge connected* if the removal of any $(k - 1)$ or fewer edges leaves G connected. If, in addition, the removal of any $(k - 1)$ or fewer vertices leaves the remaining vertices of G connected, then G is called *k-vertex connected*. We note that the *degenerate* graph consisting of a single vertex is k-edge and k-vertex connected for all values of k. A variation of Menger's Theorem states that a nondegenerate graph G is k-edge (respectively, k-vertex) connected if and only if there are k edge (respectively, vertex) disjoint paths between every pair of vertices in G.

Hence we obtain the following problem, k-connected network design with triangle inequality: given a complete graph with edge weights that satisfy the triangle inequality, and an integer k, find a minimum-weight k-edge (or k-vertex) connected spanning subgraph. We remark that for any $k \geqq 2$ this problem is NP-Hard, as the Hamiltonian Cycle problem can be reduced to a 2-connected network design problem with triangle inequality. Further, in general there will be a difference between the "edge-connected" and "vertex-connected" versions of this problem.

In the following, the word "spanning" will be omitted, for convenience. A *solution* will be a k-connected subgraph. An *optimal* subgraph or solution will be a solution of least total weight.

This paper presents some strong structural properties that optimal subgraphs can be assumed to satisfy. In particular, our results show that there are optimal subgraphs

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that are essentially edge-minimal (least number of edges) from among all k-connected subgraphs, without regard to weight (the number of edges will be within a small constant factor from being best possible). In order to describe the results in detail, we need some terminology and background.

We let V denote a set of *vertices*. A nonnegative symmetric *weight function* $d(\cdot, \cdot)$ is defined on all pairs of vertices so that the triangle inequality holds, i.e., $d(u, v) \geqq 0$, $d(u, u) = 0$, $d(u, v) = d(v, u)$, and $d(u, v) + d(v, w) \leqq d(u, w)$ for all u, v, and w in V.

Given a connected graph $G = (V, E)$, the *canonical weight function* defined by $G = (V, E)$, is given by $d(u, v) =$ minimum number of edges in a shortest path from u to v in G for all u and v in V. It is clear that this choice of weights satisfies the triangle inequality.

A recent paper [MMP] studies this network design problem in the particular case where $k = 2$. They show that for $k = 2$ the class of minimum-weight k-edge (respectively, k-vertex) connected subgraphs can be restricted to the class of k-edge (respectively, k-vertex) connected subgraphs $G = (V, E)$ satisfying the following conditions:

(I) Every vertex of G has degree k or $k + 1$;

(II) Removing any 1, 2, $\cdots$, or k edges in G does not leave all the resultant connected components all k-edge (respectively, k-vertex) connected.

They also show that any solution $G = (V, E)$ satisfying (I) and (II) for $k = 2$, is the *unique* optimal subgraph for the canonical weight function defined by G. They conjecture that these results would extend to any $k > 2$.

We note that for $k = 2$, conditions (I) and (II) ensure that the graph will be two-vertex connected. Hence, the cases for two-edge and two-vertex connectivity are in fact just one case. For $k \geqq 3$, a minimum-weight k-edge connected subgraph can have a value strictly less than a minimum-weight k-vertex connected subgraph. For example, consider the 3-edge connected graph G in Fig. 1 with the canonical weight function defined by G.

In § 2, we consider the k-edge connected problem and show that some optimal solution satisfies conditions (I) and (II). We also show that conditions (I) and (II) do not characterize the class of minimum-weight k-edge connected subgraphs for $k \geqq 3$, contrary to the case for $k = 2$.

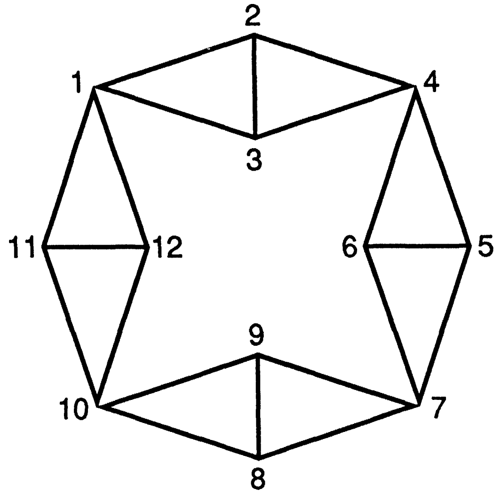

FIG. 1

In § 3, we consider the k-vertex connected case and show that some minimum-weight solution satisfies condition (II), and that it also satisfies condition (I) when $|V| \geqq 2k$. This restriction is tight, since for every $k \geqq 4$, we provide a k-vertex connected graph $G = (V, E)$ with $|V| = 2k - 1$, and maximum vertex degree of $(2k - 2)$, which is the unique optimal solution for the canonical weight function defined on G.

Our structural results are of use towards obtaining heuristics. In particular, the proof of the degree condition (I) yields a polynomial-time algorithm that, given a solution, will produce a new one that satisfies (I) without increasing cost. This is useful, as in practice it is very desirable to produce networks with vertices of small degree, and in a k-connected graph every vertex has degree at least k. Another way of restating these facts is the following: condition (I) shows that (except for small cases in the vertex-connected case) there are optimal solutions whose total number of edges is within a factor of $1 + 1/k$ of the least possible number of edges in any k-connected subgraph, independent of edge-weights. We remark that the structural properties derived in [MMP] for $k = 2$ have been used to obtain heuristics for producing low-cost networks for "real-world" problems in [MS].

In § 4, we present a polynomial-time heuristic that for each fixed $k \geqq 2$ produces a solution at most a constant factor (depending on k) larger than the optimal value.

2. Case of edge-connectivity. In this section, we show that some optimal k-edge connected subgraph satisfies conditions (I) and (II). We also show that these conditions do *not* characterize the class of minimum-weight k-edge connected subgraphs for $k \geqq 3$, contrary to the case for $k = 2$. The following definitions are due to Mader [Ma]. Let $G = (V, E)$ be a graph, with $x_i \in V$ for $0 \leqq i \leqq 2$, and with x_1 and x_2 adjacent to x_0. Then the graph G' obtained from G by deleting (x_0, x_j) $j = 1, 2$, and adding (x_1, x_2) is called a *lifting at* x_o of G. If the edge-connectedness of G', between any two vertices of $G' - x_0$, is not smaller than that of G, then the lifting is called *admissible*.

THEOREM 1. *For any set of vertices V with nonnegative symmetric weight function $d(\cdot, \cdot)$ satisfying the triangle inequality, and any $k \geqq 2$, there exists a minimum-weight k-edge connected subgraph $G = (V, E)$ satisfying conditions* (I) *and* (II).

Proof of (I). Mader [Ma] proved the following results for a graph $H = (V, E)$ with $w \in V$:

(1) If the degree of w is at least 4, and w is not cutvertex, then H has an admissible lifting at w.

(2) If w is a cutvertex, but no single edge incident to w is a cutset, then H has an admissible lifting at w. Hence, suppose that x is a vertex of G with degree at least $k + 2$. Since $k \geqq 2$, if x is not a cutvertex, then by (1) G has an admissible lifting. If x is a cutvertex, then there are at least $k \geqq 2$ edges connecting x to each connected component of $G - x$, and all edges incident to x are accounted for in this way. Hence, no such edge can be a cutset of G, and (2) applies. Thus, let G' be an admissible lifting at x.

Now G' is k-edge connected between any two vertices y and z of $G' - x$. Since, in G', the degree of x is at least k, we conclude that G' itself is k-edge connected. Further, by the triangle inequality, the weight of G' is no greater than that of G. Consequently, after repeating the above lifting procedure a finite number of times, we will obtain a minimum-weight k-edge connected graph with all degrees equal to k or $k + 1$. □

Proof of (II). Let $G = (V, E)$ be a minimum-weight k-edge connected subgraph, where G has the minimum number of edges among all minimum-weight solutions. We suppose, in order to obtain a contradiction, that (II) does not hold; that is, removing 1, 2, $\cdots$, or k edges leaves the resultant connected components k-edge connected.

Since G is k-edge connected, removing fewer than k edges from G leaves the resultant graph connected; if this resultant graph is still k-edge connected then we get a contradiction to the minimality of G. The same contradiction occurs if removing k edges from G leaves the graph connected. So, assume that the removal of the k edges (u_i, w_i) $1 \leqq i \leqq k$ results in two k-edge connected components on vertex sets U and W, with u_i in U, and w_i in W for $1 \leqq i \leqq k$. Let x_i be a neighbor of u_i in U, and y_i be a neighbor of w_i in W. The vertices u_i and x_i for $1 \leqq i \leqq k$ need not be distinct, and the vertices w_i and y_i for $1 \leqq i \leqq k$ need not be distinct. However, at least one of U, W must be nondegenerate, and consequently it must contain at least $k + 1$ vertices. Suppose W is nondegenerate. Then we may assume that $w_1 \neq w_2$, $w_1 \neq y_1$, and $w_2 \neq y_2$, and that y_i, $i = 1, 2$, is not an endpoint of any of the deleted edges.

By the triangle inequality, $d(x_1, u_1) + d(u_1, w_1) + d(w_1, y_1) \geqq d(x_1, y_1)$ and $d(x_2, u_2) + d(u_2, w_2) + d(w_2, y_2) \geqq d(x_2, y_2)$. Without loss of generality, $d(u_2, w_2) \geqq d(u_1, w_1)$, and thus replacing edges (x_1, u_1), (w_1, y_1), and (u_2, w_2) by edge (x_1, y_1) does not increase the cost of the graph, but decreases the number of edges. To obtain the desired contradiction, it remains to be shown that the resulting graph is k-edge connected.

Consider two distinct vertices s and t in U. Since the subgraph of G induced by U is assumed to be k-edge connected, there were k edge-disjoint paths between s and t before the transformation cited above was performed. If none of these paths use the deleted edge (x_1, u_1), then these same paths suffice. If one path uses the edge (x_1, u_1), replace (x_1, u_1) in that path by the added edge (x_1, y_1), a path from y_1 to w_1 in W, and the edge (w_1, u_1) to obtain k edge-disjoint paths in the transformed graph. A similar argument holds if both s and t are in W. Finally, suppose that s is in U and t is in W. Since the subgraph of G induced by U is k-edge connected even after removing the edge (x_1, u_1), there must be k edge-disjoint paths from s to $u_1, x_1, u_3, u_4, \cdots, u_k$ in U. Similarly, there are k edge-disjoint paths from t to $w_1, y_1, w_3, w_4, \cdots, w_k$ in W even after removing the edge (w_1, y_1). Together with the added edge (x_1, y_1), and existing edges $(u_1, w_1), (u_3, w_3), \cdots, (u_k, w_k)$, these paths provide k edge-disjoint paths from s to t. Hence, the transformed graph is a minimum-weight k-edge connected graph with fewer edges than G', a contradiction. □

We note that the graph G in Fig. 1 is 3-edge connected and satisfies conditions (I) and (II). Now, G is not the unique minimum-weight solution for any set of weights satisfying the triangle inequality. This follows since we may assume, without loss of generality, that $d(1, 3) \geqq d(3, 4)$ by symmetry and that $d(3, 4) + d(4, 6) \geqq d(3, 6)$ by the triangle inequality. Hence, removing the edges (1, 3) and (4, 6) and adding the edge (3, 6) leaves the graph 3-edge connected but does not increase the cost. Therefore, conditions (I) and (II) do *not* characterize the class of minimum-weight solutions for $k = 3$ as they do in the case $k = 2$.

3. Case of vertex connectivity. In this section, we analyze the design problem in the vertex-connectivity case. This section will be divided into two parts. In the first part, we give a structural theorem that is an analogue of Theorem 1. This theorem uses a technical theorem, which is proved in the second part.

3.1. The structural theorem. We will prove the following result below.

THEOREM 2. *For any set of vertices V with nonnegative symmetric weight function $d(\cdot, \cdot)$ satisfying the triangle inequality, and any $k \geqq 2$, there exists a minimum-weight k-vertex connected subgraph satisfying* (II), *and also satisfying* (I) *if* $|V| \geqq 2k$.

Before proving this theorem, let us review the implications of its statement. At first glance, the additional boundary condition imposed in (I) might seem odd. We note, however, that this requirement is needed (and tight) since for any $k \geqq 4$, there is a graph which is the unique minimum-weight k-vertex connected solution for the canonical weight function defined by G. We demonstrate this fact next. Consider a complete bipartite graph with vertex set $V_1 \cup V_2$ where $|V_1| = |V_2| = k-1$. Add a vertex x that is adjacent to every other vertex. Call this graph $G = (V, E)$. G is k-vertex connected with $|V| = 2k-1$ and the degree of x equal to $(2k-2)$. To see that G is the unique minimum-weight solution for the canonical weight function on G, note that all edges in G have weight one and all edges not in G have weight two. Now, G has (k^2-1) edges and any k-vertex connected graph has at least $(2k^2 - k/2)$ edges. Therefore, at most $p \leqq (k-2)/2$ edges of weight one can be removed from G and at most $p/2$ edges of weight two can be added to G to obtain another minimum-weight k-vertex connected solution. Since all the vertices other than x have degree equal to k, the removed edges must all be adjacent to vertex x; if not, then some vertex in the resultant graph will have degree less than k. Since $V_1 \cup V_2$ form a complete bipartite graph, the edges of weight two added to G must be entirely contained in V_1 or V_2. Therefore, the new graph is not k-vertex connected if any $p \geqq 1$ edges of weight one are removed. Therefore, G is the unique minimum-weight k-vertex connected solution, as desired.

Informally, we can explain why the additional condition in (I) arises, as follows. In order to prove condition (I) for the vertex connected case, it would be useful to have available a result on liftings similar to the theorem of Mader, used in § 2 for the edge-connectivity case, but instead for vertex connectivity. We would only need to show that if G is k-vertex connected for $k \geqq 2$ and x a vertex of G with $d(x) \geqq k+2$, then there is a lifting at x that is k-vertex connected. However, this result is not true. Consider the graph consisting of $k+2$ copies, $D_1, \cdots, D_{k+2}$, of the complete graph on k vertices, together with k additional vertices $x_1, \cdots, x_k$. Add $k(k+2)$ edges so that there is a perfect matching between $x_1, \cdots, x_k$ and each of $D_1, \cdots, D_{k+2}$, i.e., each x_i is adjacent to exactly one vertex in each D_j and each vertex $y \in D_j$ is adjacent to exactly one of $x_1, \cdots, x_k$. Then for any lifting at x_1, the vertices $x_2, \cdots, x_k$ form a cut set.

The next technical theorem is the result we need for proving the revised condition (I). Essentially, it shows that we can always find either a single lifting, or a pair of liftings that will keep the graph k-vertex connected.

THEOREM 3. *Let $G = (V, E)$ be a minimal k-vertex connected graph with $|V| \geqq 2k$ and $k \geqq 2$. If $x \in V$ has degree at least $k+2$ then either*:

(i) *there exists a lifting of x that is k-vertex connected*, *or*

(ii) *there exists a vertex $y \in V$ such that for any lifting G' at x, there exists a lifting at y of G' that is k-vertex connected.*

The requirement on the number of vertices arises naturally in the proof of Theorem 3, which shall be given in § 3.2.

Now we pass to the proof of Theorem 2.

Proof of (I). As in the proof of Theorem 1, this follows easily from Theorem 3 since, by the triangle inequality, the weight of G does not increase as a result of a lifting. □

Proof of (II). Let $G = (V, E)$ be a minimum-weight k-vertex connected subgraph V, where G has the minimum number of edges among all minimum-weight solutions. We suppose, in order to obtain a contradiction, that (II) does not hold. That is, removing $1, 2, \cdots$, or k edges leaves the resultant connected components k-vertex connected.

Since G is k-vertex connected, removing fewer than k edges from G leaves the resultant graph connected; if this resultant graph is still k-vertex connected then we get a contradiction to the minimality of G. The same contradiction occurs if removing k edges leaves the graph connected. So we assume that the removal of the k edges (u_i, w_i) $1 \leqq i \leqq k$ results in two k-vertex connected components on vertex set U and W with u_i in U and w_i in W for $1 \leqq i \leqq k$. We claim that either $|U| = 1$, $|W| = 1$ or all of the u_i and w_i are distinct. This follows since if $|U| > 1$, $|W| > 1$ and $u_1 = u_2$, then $|U| \geqq k+1$ and $|W| \geqq k+1$ since the subgraphs of G induced by U and W, respectively, are k-vertex connected, and so $\{u_1, w_3, w_4, \cdots, w_k\}$ is a $(k-1)$-vertex separator of G.

Consider the case where $|U| = 1$; the proof for $|W| = 1$ is similar. Let u be the single vertex in U which is the endpoint of all of the k removed edges. Since G has no parallel edges, each w_i is distinct for $1 \leqq i \leqq k$. Also, since the subgraph of G induced by W is k-vertex connected, $|W| \geqq k+1$. Let y_i be a neighbor of w_i in W. The y_i's need not be distinct. The triangle inequality implies that $d(u, w_i) + d(w_i, y_i) \geqq d(u, y_i)$ for $1 \leqq i \leqq k$. So, without loss of generality, we may replace edges (w_1, y_1) and (u, w_2) by the edge (u, y_1) without increasing the cost of the graph. We claim that the resultant graph is still k-vertex connected, which would provide the desired contradiction. Consider u in U and any t in W. Since the subgraph of G induced by W is k-vertex connected, there are k vertex-disjoint paths from t to vertices $w_1, y_1, w_3, w_4, \cdots, w_k$ even excluding the removed edge (w_1, y_1). Continuing these paths with the added edge (u, y_1) and existing edges $(u, w_1), (u, w_3), (u, w_4), \cdots, (u, w_k)$ yields the desired result.

Now, consider the case where $|U| \geqq k+1$, $|W| \geqq k+1$ and the vertices u_i and w_i are all distinct for $1 \leqq i \leqq k$. Let x_i be a neighbor of u_i in U, and let y_i be a neighbor of w_i in W. The x_i's need not be distinct, and the y_i's need not be distinct. By the triangle inequality, $d(x_i, u_i) + d(u_i, w_i) + d(w_i, y_i) \geqq d(x_i, y_i)$ for $1 \leqq i \leqq k$. So, without loss of generality, replacing edges (x_1, u_1), (w_1, y_1) and (u_2, w_2) by the edge (x_1, y_1) does not increase the total cost. We need only show that the resultant graph is k-vertex connected to obtain the desired contradiction.

First, consider distinct vertices s and t both in U; the case where both s and t are in W is similar. Since the subgraph of G induced by U is k-vertex connected, there were k vertex-disjoint paths from s to t in U. If none of these paths used the deleted edge (x_1, u_1) then we are done. If one path used this edge then replace the removed edge (x_1, u_1) by the added edge (x_1, y_1), a path in W from y_1 to w_1, and the edge (u_1, w_1) to complete the path.

Finally, consider a vertex s in U, and a vertex t in W. Since the subgraph of G induced by U is k-vertex connected, there exists k vertex-disjoint paths from s to $u_1, x_1, u_3, \cdots, u_k$ not using edge (x_1, u_1). Similarly, there are k vertex-disjoint paths from t to $w_1, y_1, w_3, \cdots, w_k$ not using edge (y_1, w_1). Combining these paths with the added edge (x_1, y_1) and existing edges $(u_1, w_1), (u_3, w_3), \cdots, (u_k, w_k)$ yields the desired result. Hence, the resultant graph is a minimum-weight k-vertex connected subgraph with fewer edges than G, a contradiction. □

3.2. Proof of Theorem 3. Below we will prove the theorem on liftings which was used in the proof of Theorem 2. First we need some auxiliary definitions and results.

For a graph $G = (V, E)$ and a set of vertices $W \subseteq V$, let $\delta_G(W)$ be the set of vertices contained in $V \backslash W$ that is adjacent to at least one vertex in W. It is not hard to prove for $W, U \subseteq V$, that the following submodular inequality holds:

$$|\delta(W \cap U)| + |\delta(W \cup U)| \leqq |\delta(W)| + |\delta(U)|.$$

Below we will require the following slightly stronger result that involves two graphs defined on the same vertex set.

PROPOSITION 1. *Let $G = (V, E)$ and $G' = (V, E')$ be graphs. Let $W, U \subseteq V$. Suppose that $\delta_G(W \cap U) \subseteq \delta_{G'}(W \cap U)$ and $\delta_{G'}(W \cup U) \subseteq \delta_G(W \cup U)$. Then:*

(i) $|\delta_G(W \cap U)| + |\delta_{G'}(W \cup U)| \leqq |\delta_G(W)| + |\delta_{G'}(U)|$, *and*

(ii) *if, in addition, G and G' are k-vertex connected and $|\delta_G(W)| = |\delta_{G'}(U)| = k$, $W \cap U \neq \varnothing$, and $|W \cup U| \leqq |V| - k$ then $|\delta_G(W \cup U)| = |\delta_{G'}(W \cup U)| = k$.*

Proof. Since $|X| + |Y| = |X \cup Y| + |X \cap Y|$, we can prove (i) by proving that $\delta_G(W \cap U) \cap \delta_{G'}(W \cup U) \subseteq \delta_G(W) \cap \delta_{G'}(U)$, and $\delta_G(W \cap U) \cup \delta_{G'}(W \cup U) \subseteq \delta_G(W) \cup \delta_{G'}(U)$. To prove the first condition suppose $y \in \delta_G(W \cap U) \cap \delta_{G'}(W \cup U)$. Then $y \notin W \cup U$ and is in $\delta_G(W \cap U)$. So y is adjacent in G to a vertex in $W \cap U$. Thus $y \in \delta_G(W)$. Also $\delta_G(W \cap U) \subseteq \delta_{G'}(W \cap U)$ so $y \in \delta_{G'}(U)$. The proof of the latter condition is similar.

To prove (ii), observe that either $|W \cap U| = |V| - k$ or $\delta_G(W \cap U)$ is a cutset in G. So $|\delta_G(W \cap U)| \geqq k$. Also, either $\delta_{G'}(W \cup U) = V \backslash (W \cup U)$ or $\delta_{G'}(W \cup U)$ is a cutset in G'. So $|\delta_{G'}(W \cup U)| \geqq k$. So (ii) follows from (i). □

We will say that a set $W \subseteq V$ is a (k, T, x) *separator* in G if $\varnothing \neq T \subseteq W$, $x \notin W \cup \delta(W)$ and $|\delta(W)| = k$.

PROPOSITION 2. *Let G be a k-vertex connected graph. Let $x \in V$ and $T \subseteq V$. If there exists a (k, T, x) separator in G, then there exists a unique maximal (k, T, x) separator in G.*

Proof. If W and W' are both (k, T, x) separators then $T \subseteq W \cap W'$ and $x \notin W \cup W' \cup \delta(W) \cup \delta(W')$. So by Proposition 1, $W \cup W'$ is a (k, T, x)-separator. Thus $W = \cup\{Z \mid Z$ is a (k, T, x)-separator$\}$ is a maximal (k, T, x)-separator. □

Next we prove Theorem 3, which we restate here.

THEOREM 3. *Let $G = (V, E)$ be a minimal k-vertex connected graph with $|V| \geqq 2k$ and $k \geqq 2$. If $x \in V$ has degree at least $k + 2$ then either:*

(i) *there exists a lifting of x that is k-vertex connected, or*

(ii) *there exists a vertex $y \in V$ such that for any lifting G' at x, there exists a lifting at y of G' that is k-vertex connected.*

Proof. Let $x \in V$ have degree at least $k + 2$. Suppose there is no lifting at x that is k-vertex connected. We will show that condition (ii) holds.

Let d be the degree of x and let $x_1, \cdots, x_d$ be the neighbors of x. For each pair of neighbors x_i and x_j of x, let G_{ij} be the lifting at x with respect to vertices x_i and x_j. By assumption, G_{ij} is not k-vertex connected. Let S be a cutset in G_{ij} with $|S| \leqq k - 1$. Let W be a connected component of the graph $G_{ij} \backslash S$. Since S is not a cutset in G, W contains at least one of the vertices x, x_i or x_j. So $G_{ij} \backslash S$ contains exactly two connected components W and U. Without loss of generality, $x \in U$, $\{x_i, x_j\} \subseteq W \cup S$, and $x_i \in W$. Since x has at least k neighbors in G_{ij}, $|U| \geqq 2$. $S \cup \{x\}$ is a cutset in G, so $|S| = k - 1$. If there exists a $(k - 1, x_i, x)$ separator in G_{ij}, let W_{ij} be a maximal $(k - 1, x_i, x)$ separator in G_{ij}. If there is no $(k - 1, x_i, x)$ separator in G_{ij}, then define $W_{ij} = \varnothing$. Note that if $x_j \in W_{ij}$, then $W_{ij} = W_{ji}$. Let $S_{ij} = \delta_{G_{ij}}(W_{ij})$. Let $U_{ij} = V \backslash (W_{ij} \cup S_{ij})$.

Similarly, for each neighbor x_i of x, let G_i be the graph G with the edge (x, x_i) deleted. Since G is minimal k-vertex connected, G_i is not k-vertex connected, but it is $(k - 1)$ vertex connected, and there exists a $(k - 1, x_i, x)$ separator in G_i. Let W_i be the maximal $(k - 1, x_i, x)$ separator in G_i. Let $S_i = \delta_{G_i}(W_i)$ and let $U_i = V \backslash (W_i \cup S_i)$.

The remainder of the proof will be broken into several steps.

Step 3.1. For any two neighbors x_i and x_j of x, $W_i \cap W_j = \varnothing$. If $W_{ij} \neq \varnothing$ then $W_i \subseteq W_{ij}$ and if, in addition, $x_j \notin W_{ij}$, then $W_i = W_{ij}$.

Proof. The only edge that G_i contains that is not an edge of G_j is (x, x_j), and the only edge that G_j contains that might not be an edge of G_i is the edge (x, x_i). Thus, it is easy to verify that $\delta_{G_i}(W_i \cap W_j) \subseteq \delta_{G_j}(W_i \cap W_j)$, and $\delta_{G_j}(W_i \cup W_j) \subseteq \delta_{G_i}(W_i \cup W_j)$.

Now x and all neighbors of x except for x_i and x_j are not in $W_i \cup W_j$. So $|W_i \cup W_j| \leqq |V| - (k+1)$. If $W_i \cap W_j \neq \emptyset$, then by Proposition 1, $|\delta_{G_i}(W_i \cap W_j)| = k - 1$. But $x_i \notin W_i \cap W_j$, so $\delta_G(W_i \cap W_j) = \delta_{G_i}(W_i \cap W_j)$. This contradicts the fact that G is k-vertex connected and establishes the first condition.

Similarly, it is also easy to verify that if $W_{ij} \neq \emptyset$ then $\delta_{G_i}(W_i \cap W_{ij}) \subseteq \delta_{G_{ij}}(W_i \cap W_{ij})$, $\delta_{G_{ij}}(W_i \cup W_{ij}) \subseteq \delta_{G_i}(W_i \cup W_{ij})$, and $|W_i \cup W_{ij}| \leqq |V| - (k+1)$. Also, $W_i \cap W_{ij} \neq \emptyset$ since $x_i \in W_i \cap W_{ij}$. So by Proposition 1, $|\delta_{G_{ij}}(W_i \cup W_{ij})| = k - 1$. By the maximality of W_{ij}, $W_i \subseteq W_{ij}$. If $x_j \notin W_{ij}$, then W_{ij} is a $(k-1, x_i, x)$ separator in G_i. By the maximality of W_i, $W_{ij} \subseteq W_i$, which establishes the second condition. □

Step 3.2. If $W_{ij} \neq W_{ji}$ then $W_j \cap S_i \neq \emptyset$.

Proof. If $W_{ij} \neq \emptyset$ and $W_{ij} \neq W_{ji}$, then $x_j \notin W_{ij}$ and $W_i = W_{ij}$. Now $\delta_{G_i}(W_i) \subseteq \delta_{G_{ij}}(W_i)$ and $|\delta_{G_i}(W_i)| = k - 1 = |\delta_{G_{ij}}(W_i)|$ imply that $\delta_{G_i}(W_i) = \delta_{G_{ij}}(W_i)$. Consequently, since $x_j \in \delta_{G_{ij}}(W_i)$, we have $x_j \in \delta_{G_i}(W_i)$.

If $W_{ij} \neq \emptyset$, then $W_{ji} \neq \emptyset$ and $x_i \in \delta_{G_j}(W_j)$. There exists a path in G_j from x_j to x_i with all vertices of the path except x_i contained in W_j. Since $x_j \notin W_i$, there must be a vertex on this path contained in S_i. □

In reading the following, it will be helpful to consider the diagram in Fig. 2. This is a partition of the vertices of G into nine disjoint sets. Two of the sets in Fig. 2 are said to be adjacent if their boundaries in Fig. 2 share a point or a line segment in the diagram. (Note that a set is adjacent to itself.) Then two vertices in G can be adjacent only if the two sets that contain them are adjacent in Fig. 2. □

Step 3.3. If $W_{ij} \neq \emptyset$ and $W_i \neq \emptyset$, then $W_i = W_{ij} \cap W_i$, and $S_i = (S_{ij} \cap S_i) \cup (S_i \cap W_{ij}) \cup (S_{ij} \cap W_i)$.

Proof. Clearly, $x_i \in W_{ij} \cap W_i$, and x together with at least $(k-1)$ neighbors of x are not in $W_{ij} \cup W_i$. Hence by Proposition 1, $|\delta_G(W_{ij} \cap W_i)| = k = |\delta_G(W_{ij} \cup W_i)|$.

Now $x \in \delta_G(W_{ij} \cap W_i)$ but $x \notin \delta_{G_i}(W_{ij} \cap W_i)$. So $W_{ij} \cap W_i$ is a $(k-1, x_i, x)$ separator in G_i. By Step 3.1 and the maximality of W_i, $W_i = W_{ij} \cap W_i$.

Let $T = (S_{ij} \cap S_i) \cup (S_{ij} \cap U_i) \cup (S_i \cap U_{ij})$, and let $T' = (S_{ij} \cap S_i) \cup (S_{ij} \cap W_i) \cup (S_i \cap W_{ij})$. Now, $|T| + |T'| = |S_{ij}| + |S_i|$. Also, $\delta_G(W_{ij} \cap W_i) \subseteq \{x\} \cup T'$ and $\delta_G(W_{ij} \cup W_i) \subseteq \{x\} \cup T$. So $|T| = |T'| = k - 1$, and $\delta_G(W_{ij} \cap W_i) = \{x\} \cup T'$. Hence, $S_i = (S_{ij} \cap S_i) \cup (S_{ij} \cap W_i) \cup (S_i \cap W_{ij})$. □

Step 3.4. If $W_j \cap S_i = \emptyset$ then $S_i = S_j = S_{ij}$.

Proof. From Steps 3.2 and 3.1, and since $|S_i| = k - 1$, we see that there is another neighbor x of x such that $W \cap S_i = \emptyset$. By Step 3.2, we get $x_j \in W_{ij}$. But $x_j \notin W_{ij} \cap$

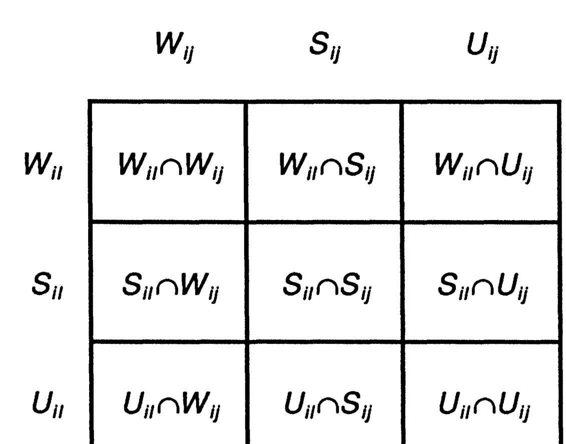

FIG. 2

W_i by Step 3.3, and $x_j \notin W_{ij} \cap S_i$. Since $x_j \notin S_i$, we have $x_j \in W_{ij} \cap U_i$. Similarly, $x \in W_i \cap U_{ij}$.

Let $R = (S_{ij} \cap S_i) \cup (S_{ij} \cap U_i) \cup (S_i \cap W_{ij})$, and let $R' = (S_{ij} \cap S_i) \cup (S_i \cap U_{ij}) \cup (S_{ij} \cap W_i)$. Now $|R| + |R'| = |S_{ij}| + |S_i|$. Also, $\delta_G(W_{ij} \cap U_i) \subseteq \{x\} \cup R$, and $\delta_G(W_i \cap U_{ij}) \subseteq \{x\} \cup R'$. Hence, $|R| = |R'| = k - 1$. So $W_{ij} \cap U_i$ is a $(k-1, x_j, x)$ separator in G_j. Since $W_j \subseteq W_{ij}$, $W_j \cap W_{ij} \cap W_i = \varnothing$ and $W_j \cap W_{ij} \cap S_i = \varnothing$, we have $W_j \subseteq W_{ij} \cap U_i$. By the maximality of W_j, we get $W_j = W_{ij} \cap U_i$ and $S_j = R$. Similarly, $W = W_i \cap U_{ij}$ and $S = R'$. Thus, $S_j \cap S = S_i \cap S_j \cap S$. We also get that $W_a \cap S_b = \varnothing$ for $\{a, b\} \subseteq \{i, j, k\}$. So from $S_i \cap S_j = S_i \cap S_j \cap S$, we get $S_i \cap W_{ij} = \varnothing$. Similarly, $S_{ij} \cap W_{ie} = \varnothing$. Thus, $S_i = S_{ij} \cap S_i$. So $|S_{ij} \cap S_i| = k - 1$ and $S_j = S_{ij} \cap S_i = S_i = S_{ij}$. □

Step 3.5. $W_{ij} = W_{ji}$ if and only if $x_j \notin S_i$.

Proof. Suppose $W_{ij} = W_{ji}$. Let x be a vertex such that $W \cap S_i = \varnothing$. By Steps 3.3 and 3.4, we have $S_i = (S_i \cap W_{ij}) \cup (S_{ij} \cap S_i) \cup (S_{ij} \cap W_i) = S_i$. So $S_{ij} \cap W_i = \varnothing = S_i \cap U_{ij}$. Thus, $\delta_G(W_i \cap U_{ij}) = (S_{ij} \cap S_i) \cup \{x\}$, and $|S_{ij} \cap S_i| \geqq k - 1$. So $S_{ij} = S_{ij} \cap S_i = S_i = S_i$ and $x_j \notin S_i$.

Suppose $x_j \notin S_i$. Let X be the set of neighbors of x such that $W \cap S_i = \varnothing$ for each $x \in X$. So $S_i = S$ for each $x \in X$. Now if $W \cap S_j = \varnothing$, then $S_j = S = S_i$ and $W_{ij} = W_{ji}$ by Step 3.2. Hence we may assume that $W \cap S_j \neq \varnothing$ for each $x \in X$. There exist disjoint paths from x_j to the vertices in S_j with each path contained in $W_j \cup S_j$. For each $x \in X$ and for each vertex y in $W \cap S_j$, the path from x_j to y must contain a vertex of $S = S_i$. So $|S_i \cap W_j| \geqq |X|$. But for each neighbor $x_g \notin X$, $|S_i \cap W_g| \geqq |X|$. So $|S_i| \geqq d(x) - 1$. □

Step 3.6. Let x_{i_0} be a neighbor of x. Let $x_1, \cdots, x_t$ be the neighbors of x that are not in S_{i_0}. Then $V = \{x\} \cup S_{i_0} \cup_{i=1}^{t} W_i$.

Proof. Let G' be the graph G induced on $V \backslash S_{i_0}$ with edges (x, x_i) deleted for $1 \leqq i \leqq t$. By Step 3.5, x is an isolated vertex in G'. So if $y \in V$ but $y \notin \{x\} \cup S_{i_o} \cup \cup_{i=1}^{t} W_i$, then S_{i_0} separates y from x in G. □

Step 3.7. Let x_i be a neighbor of x. Then S_i contains no neighbors of x.

Proof. Suppose $x_j \in S_i$. Let $x_{i_1}, \cdots, x_{i_t}$ be the neighbors of x that are not in S_i. Let $x_{j_1}, \cdots, x_{j_s}$ be the neighbors of x that are not in S_j. Then $V = \{x\} \cup S_i \cup_{i=1}^{t} W_i = \{x\} \cup S_j \cup_{j=1}^{s} W_j$.

By Step 3.5, $\{x_{j_1}, \cdots, x_{j_s}\} \cap \{x_{i_1}, \cdots, x_{i_t}\} = \varnothing$. So $V \subseteq \{x\} \cup S_i \cup S_j$. Hence $|V| \leqq 2k - 1$. □

We are now ready to complete the proof of Theorem 3. By Steps 3.5 and 3.7, we have $S_i = S_j$ for any two neighbors x_i, and x_j of x. So let $S = S_i$, and let $y \in S$.

Let y_i be a vertex W_i that is adjacent to y. Given two neighbors, x_1 and x_2 of x, let G' be the lifting at y of G_{12} that is obtained by deleting edges (y, y_2) and (y, y_3) and adding edge (y_2, y_3). We can now finish the proof of Theorem 3.1.

Step 3.8. G' is k-connected.

Proof. Since G is k-vertex connected, we need only show that there are k vertex disjoint paths in G' between each of the pairs (x, x_1), (x, x_2), (y, y_2), and (y, y_3). In the remainder of the proof, we will say that a path P is contained in a vertex set $Y \subseteq V$ if all vertices of P except for the endpoints are in Y. We will say that a set of paths $P_1, \cdots, P_t$ are paths from u to $u_1, \cdots, u_t$ if path P_i is a path from u to u_i. Let $z_1, \cdots, z_{k-1}$ be the vertices in S with $y = z_1$.

For (x, x_1) let $P_1, \cdots, P_{k-1}$ be vertex disjoint paths in G' contained in W_1 from x_1 to $z_1, \cdots, z_{k-1}$. Extend P_i along a path in W_{i+3} to x. Let P_k be a path x_1 to x_2 extended along a path to y_2 contained in W_2, then to y_3 and then along a path in W_3 to x.

For (x, x_2) let $P_1, \cdots, P_k$ be vertex disjoint paths in G from x_2 to $z_1, \cdots, z_{k-1}$, y_2. If one of these paths contains the vertex x, call that path P. P cannot be the path from x_2 to y_2. All paths except P are paths in G'. Extend P_k in G' to y_3 and then along a path in W_3 to x. For $P_i \neq P$, extend P_i along a path in W_{i+3} to x. If z_j is an endpoint of P, then truncate P at x_2 and extend it in G' to x and then to z_j in W_1 and then along a path in W_{j+3} to x.

For (y, y_2), let $P_1, \cdots, P_k$ be vertex disjoint paths in G contained in W_2 from y_2 to $z_1, \cdots, z_{k-1}, x_2$. All paths except P_1 are paths in G'. Replace P_1 with a path in G' starting at y_2, going to y_3, then to x along a path in W_3, and then to y along a path in W_4. For $2 \leqq i \leqq k - 1$, extend P_i to y_{i+3} along a path in W_{i+3} and then to y. Extend P_k to x, and then to y along a path in W_1.

For (y, y_3) let $P_1, \cdots, P_k$ be vertex disjoint paths in G contained in W_3 from y_3 to $z_1, \cdots, z_{k-1}, x$. All paths except P_1 are paths in G'. Replace P_1 with a path in G' starting at y_3, going to y_2, then to x_2 along a path in W_2, then to x_1, and then to y along a path in W_1. For $2 \leqq i \leqq k - 1$, extend P_i to y along a path in W_{i+2}.

This completes the proof of Theorem 3. $\square$

4. Concluding remarks. We have derived conditions on the class of minimum-weight k-edge (or k-vertex) connected networks, where the distances between the points satisfy the triangle inequality. This generalizes recent results [MMP] for the case $k = 2$. We also showed that these conditions do not characterize the optimal solutions for $k \geqq 3$, contrary to the case $k = 2$. We leave as an open question the problem of determining additional properties which characterize these classes.

For completeness, we mention that for each fixed $k \geqq 2$, there is a polynomial-time algorithm that produces a solution of cost at most a constant factor (depending on k) of the optimum. This is obtained as follows: start with a Hamiltonian cycle C produced by the Christofides heuristic. It is well known that the value of C is at most $\frac{3}{2}$ that of an optimal 2-connected spanning subgraph [FJ]. Next, let C^k denote the graph obtained from C by adding an edge to join any two vertices that are within distance k in C (distance in the graph-theoretic sense). It is not difficult to see that C^k is $2k$-connected, and that its value is at most $3k(k+1)/4$ times that of an optimal 2-connected spanning subgraph. Since the cost of *any* k-connected spanning subgraph is at least that of an optimal 2-connected spanning subgraph, we obtain the desired result.

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